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TREATMENT OF THE SOLIDIFICATION PROBLEM INSIDE AND OUTSIDE CYLINDERS BY VARIABLE TIME STEP METHODS

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NOMENCLATURE

- u, temperature;
- t, dimensionless time, $4\tau/\gamma$;
- r, radial coordinate;
- x, cartesian coordinate, r^2;
- r_0, position of the solid–liquid interface in radial coordinates;
- s, position of the solid–liquid interface in cartesian coordinates, r_0^2.

Greek symbols
- β, γ, parameters;
- Δt, time step;
- Δx, space interval;
- τ, time.

Subscripts
- i, j, locations in x–t plane.

Superscripts
- k, number of iterations.

1. INTRODUCTION

THE PHENOMENON of heat conduction with change of phase due to melting/freezing occurs in many areas of current practical interest, e.g. the production or melting of ice, the solidification of castings, the preservation of foodstuffs, the penetration of frost into the earth, and the ablation of space missiles due to aerodynamic heating. As the space domain where the heat conduction equation is to be solved changes with time, these problems are known as 'moving boundary' or 'Stefan' problems [1]. The solution of these problems is sought in a changing domain whose shape is not known *a priori*. Because of this, moving boundary problems cannot be dealt with by straightforward analytical methods. A number of methods have been suggested for solving these problems [2–5]. In his excellent book, Rubinstein [6] cites a number of moving boundary problems arising in industry and elsewhere along with the solutions to many of them. The outcome of some conferences on this subject are also available [6–8].

The authors have proposed a "modified variable time step" (MVTS) method and an extended version of the method of Douglas and Gallie [9], called the EDG method, for 1-dim. problems [10–12]. In the present paper an attempt is made to apply these methods to the problem of solidification inside and outside right circular cylinders. Two sample problems are selected for analysis, the first concerned with the solidification of a saturated liquid inside a cylinder with a convective boundary condition, the second with solidification outside a cylinder under a constant temperature at the fixed surfaces. The methods presented are illustrated in detail with respect to the first problem only. The procedure for the second is similar.

The problem of inward solidification in a cylinder with a convective boundary condition has been attempted [13–17]. In the present paper it has been solved by the two "variable time step" (VTS) methods (MVTS and EDG) developed by the authors. Although, Goodling and Khader [18] have also solved this problem by their own VTS method, the present methods are more systematic and efficient. The numerical results are compared with the tabular results of Baxter [15] and Tao [16]. The second problem of outward solidification of a cylinder under a constant surface temperature is also solved by the MVTS and EDG methods. The results obtained are compared with those of Bell [19], who made use of the heat balance integral method in its refined form. The agreement is found to be very good in both the cases.

2. THE PROBLEM

The problem of inward solidification of a liquid, contained in an infinitely long circular cylinder, under a convective boundary condition may be expressed, in non-dimensional form, as

$$\gamma\frac{\partial u}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad r_0(\tau) < r < 1, \quad \tau > 0;$$

with the boundary conditions

$$-\frac{\partial u}{\partial r} = u/\beta, \quad r = 1, \quad \tau > 0;$$

$$u(r,\tau) = 1, \quad 0 \leqslant r \leqslant r_0(\tau), \quad \tau \geqslant 0;$$

$$\frac{\mathrm{d}r_0}{\mathrm{d}\tau} = \frac{\partial u}{\partial r}, \quad r = r_0(\tau), \quad \tau > 0;$$

and the initial condition

$$r_0(0) = 1,$$

where $u(r,\tau)$ denotes the temperature of the solid phase at a radial distance r from the centre; $r_0(\tau)$ is the position of the solid–liquid interface, and β and γ are parameters. It is assumed that the liquid is at its fusion temperature of unity at $t = 0$.

Changing the variables by setting $x = r^2$, $t = 4\tau/\gamma$, the above system of equations transforms to

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), \quad s(t) < x < 1, \quad t \geqslant 0; \tag{1}$$

$$-\frac{\partial u}{\partial x} = u/2\beta, \quad x = 1, \quad t > 0; \tag{2}$$

$$u(x,t) = 1, \quad 0 \leqslant x \leqslant s(t), \quad t \geqslant 0; \tag{3}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\gamma s}\frac{\mathrm{d}s}{\mathrm{d}t}, \quad x = s(t), \quad t > 0; \tag{4}$$

$$s(0) = 1, \tag{5}$$

where $s(t)$ is the corresponding distance of the interface from $x = 0$.

3. OUTLINE OF THE METHODS

In order to apply a VTS method, the region of interest in the problem, $0 \leqslant x \leqslant 1.0$, is subdivided into, say, n intervals each of width Δx with mesh points $x_0 = 0 < x_1 < \ldots < x_n = 1.0$, so that $n\Delta x = 1.0$. Consider the time interval $\Delta t_p (= t_{p+1} - t_p)$, $p = 0(1)n-1$, in which the boundary moves a distance Δx

from $x = x_{n-p}$ to $x = x_{n-(p+1)}$. Any point (x_i, t_j) in the x–t plane is given by

$$\left(i\Delta x, \sum_{m=0}^{j-1} \Delta t_m\right).$$

Assuming that $\Delta t_0, \Delta t_1, \ldots, \Delta t_{j-1}$ have already been calculated and the temperatures up to and including $t = t_j$ are also known, we wish to compute Δt_j as well as the temperatures in the medium at $t = t_{j+1}$, i.e. $u_{i,j+1}, i = n-j(1)n$, when $u_{n-(j+1),j+1} = 1$ from boundary condition (3). The details of the methods employed in the present paper are given below.

3.1. *Extension of Douglas and Gallie's* (*EDG*) *method*

Discretizing the LHS of equation (1) by backward difference and the RHS by central difference, at (x_i, t_{j+1}), we obtain

$$(u_{i,j+1}-u_{i,j})/\Delta t_j = x_i\{(u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1})/[(\Delta x)^2]\} + (u_{i+1,j+1}-u_{i-1,j+1})/(2\Delta x),$$

which yields

$$-r(2x_i-\Delta x)u_{i-1,j+1}+2(1+2rx_i)u_{i,j+1}-r(2x_i+\Delta x)u_{i+1,j+1} = 2u_{i,j}, \quad i = n-j(1)n-1 \quad (7)$$

where $r[=\Delta t_j/(\Delta x)^2]$ is different at different time steps.

In order to compute Δt_j we integrate equation (1) with respect to x from $s(t)$ to 1. Making use of boundary conditions (2)–(4), this gives

$$\frac{d}{dt}\int_{s(t)}^{1} u(x,t)\mathrm{d}x = -[(1+\gamma)/\gamma]\frac{\mathrm{d}s}{\mathrm{d}t} - u(1,t)/(2\beta).$$

Further integration with respect to t between t_j and t_{j+1} provides

$$\int_{s_{j+1}}^{1} u(x,t_{j+1})\mathrm{d}x - \int_{s_j}^{1} u(x,t_j)\mathrm{d}x = [(1+\gamma)/\gamma]\Delta x-(1/2\beta)u(x_n,t_{j+1})\Delta t_j,$$

where

$$s_j = s(t_j)$$

and

$$\Delta x = s_j - s_{j+1}.$$

Approximating the integrals by the trapezoidal rule and rearranging the terms, we get

$$\Delta t_j = (2\beta/u_{n,j+1})\{[(1+\gamma)/\gamma]\Delta x + A_j - A_{j+1}\}, \quad (8)$$

where

$$A_j = \left[\tfrac{1}{2}(u_{n-j,j}+u_{n,j}) + \sum_{i=n-(j-1)}^{n-1} u_{i,j}\right]\Delta x,$$

and

$$A_{j+1} = \left[\tfrac{1}{2}(u_{n-(j+1),j+1}+u_{n,j+1}) + \sum_{i=n-j}^{n-1} u_{i,j+1}\right]\Delta x.$$

The procedure for computing Δt_j and $u_{i,j+1}$, $i = n-j(1)n$ may be followed as described in the earlier papers [10, 11]. At the kth iteration, the relevant simultaneous equations become

$$u^{(k)}_{n-(j+1),j+1} = 1, \quad (9)$$

$$-r^{(k)}(2x_i-\Delta x)u^{(k)}_{i-1,j+1} + 2(1+2r^{(k)}x_i)u^{(k)}_{i,j+1} - r^{(k)}(2x_i+\Delta x)u^{(k)}_{i+1,j+1} = 2u_{i,j}, \quad i = n-j(1)n-1 \quad (10)$$

$$u^{(k)}_{n,j+1} = [2\beta/(2\beta+\Delta x)]u^{(k)}_{n-1,j+1} \quad (11)$$

where

$$r^{(k)} = \Delta t_j^{(k)}/[(\Delta x)^2].$$

The $(k+1)$th iteration for Δt_j can be written, from equation (8), as

$$\Delta t_j^{(k+1)} = (2\beta/u^{(k)}_{n,j+1})\{[(1+\gamma)/\gamma]\Delta x + A_j - A_{j+1}\} \quad (12)$$

where

$$A_j = \left[\tfrac{1}{2}(u_{n-j}+u_{n,j}) + \sum_{i=n-(j-1)}^{n-1} u_{i,j}\right]\Delta x,$$

and

$$A_{j+1} = \left[\tfrac{1}{2}(u^{(k)}_{n-(j+1),j+1}+u^{(k)}_{n,j+1}) + \sum_{i=n-j}^{n-1} u^{(k)}_{i,j+1}\right]\Delta x.$$

The procedure is started by choosing $\Delta t_j^{(0)}$ equal to Δt_{j-1} (calculated at the previous step).

3.2. *Modified Variable Time Step* (*MVTS*) *method*

In this method, proposed by the present authors, the simultaneous equations (9)–(11) are obtained as described in Section 3.1. However, in order to compute Δt_j we make use of the finite difference replacement of the interface condition (4), giving

$$(s_{j+1}-s_j)/\Delta t_j = -\Delta x/\Delta t_j = \tfrac{1}{2}(s_j+s_{j+1})\gamma\left.\frac{\partial u}{\partial x}\right|_{n-(j+1),j+1},$$

which, on using equation (3), provides the $(k+1)$th iteration of Δt_j,

$$\Delta t_j^{(k+1)} = 2(\Delta x)^2/[\gamma(s_j+s_{j+1})(1-u^{(k)}_{n-j,j+1})]. \quad (13)$$

In deriving expression (13) an average value of s is taken. The computing procedure is followed exactly in the same manner as in the EDG method with the value of Δt_j being modified by equation (13) instead of equation (12).

4. NUMERICAL RESULTS AND DISCUSSION

The problem defined by equations (1)–(5) is solved for $\beta = 0.1$, 0.5 and 1.0 with $\gamma = 1.0$, 2.0. The EDG and the MVTS methods are applied choosing $\Delta x = 0.05$ and 0.025, i.e. by subdividing the region $0 \leqslant x \leqslant 1.0$ into 20 and 40 equal intervals, respectively. Iterations are carried until two successive values of Δt differ by less than 10^{-5}.

In order to start the computations the expressions for Δt_0 for the EDG and the MVTS methods may be obtained from equations (12) and (13) respectively. However, for the sake of comparing the results, the starting value Δt_0 is taken from equation (12) for the MVTS method also.

Table 1 gives the time required for the complete freezing of the cylinder from both methods. Comparative figures are also given from Baxter [15] and Tao [16]. As may be seen from the table, our results compare very well with those of Tao [16].

The second problem concerns outward solidification of a cylinder when the fixed surface $x = 1$ is subjected to a constant temperature $u = 0$ for $t > 0$. It is assumed that the temperature is everywhere unity outside the cylinder at zero time. The time taken by the interface to reach various positions from the fixed surface $x = 1$ onwards, are computed with $\gamma = 1.0$ taking $\Delta x = 0.05$ and 0.025. Bell [19] deals with the same problem and computes the numerical results by three methods: (i) the integral method as suggested by Lardner and Pohle [20], (ii) the refinement of the integral method suggested by Bell [19] himself, and (iii) the isotherm migration method (IMM) of Dix and Cizek [21]. All the results are reproduced in Table 2 with our results. A close correspondence may be observed between our results and those of Bell [19].

Another VTS method has been described by Yuen and Kleinman [22]. However, this method, as well as that of Goodling and Khader [18], is not strictly iterative and some

Table 1. Comparison of time required for the complete freezing of the cylinder for inward solidification (Problem 1)

β	γ	Baxter [19]	Tao [20]	EDG method $\Delta x = 0.05$	EDG method $\Delta x = 0.025$	MVTS method $\Delta x = 0.05$	MVTS method $\Delta x = 0.025$
1.0	1.0	1.00	1.045	1.035	1.034	1.055	1.044
	2.0	1.18	1.243	1.225	1.229	1.245	1.240
0.5	1.0	0.70	0.736	0.730	0.729	0.745	0.737
	2.0	0.82	0.897	0.884	0.888	0.901	0.897
0.1	1.0	0.42	0.463	0.461	0.460	0.475	0.467
	2.0	0.52	0.580	0.567	0.573	0.586	0.583

Table 2. Comparative figures for interface positions at different times for outward solidification (Problem 2)

Time	Lardner and Pohle	Bell [22] ($n = 8$)	I.M.M.	EDG method $\Delta x = 0.05$	EDG method $\Delta x = 0.025$	MVTS method $\Delta x = 0.05$	MVTS method $\Delta x = 0.025$
0.05	1.2526	1.2638	1.2695	1.2567	1.2628	1.2603	1.2646
0.10	1.3543	1.3672	1.3769	1.3630	1.3693	1.3679	1.3718
0.50	—	—	—	1.7911	1.7991	1.8014	1.8043

nanipulations are required when using it. Also it may be ›ointed out that the EDG method, although very versatile, uffers from a drawback in that it uses all the u values for stimating Δt, while in the MVTS method only one value of u is ;ecessary to get an improved estimate of Δt.

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